



TITLE:

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The Generalized Fermat-Steiner Problem with Free Ends

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1 Introduction

Let $G = (e_1, e_2, \dots, e_n)$ be a conneted graph such that the degree of its vertices are all 3 except for the end points. In other words, G is a network with triple junctions. For a given region $\Omega \subset \mathbb{R}^2$, a set of line segment Γ_G is called *admissible* for G if Γ_G is isomorphic to G and all the end points of Γ_G are on $\partial\Omega$.

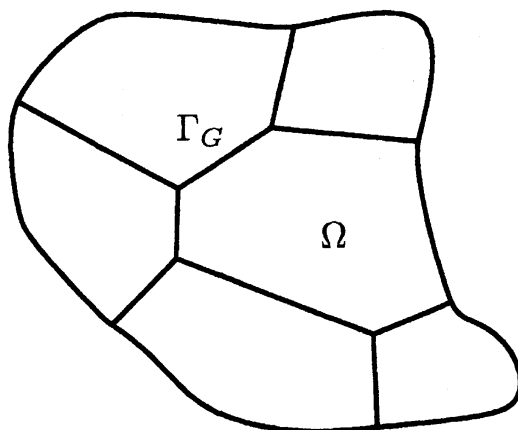


Figure 1: An example of Γ_G .

We assign a positive number σ_i to each edge e_i , which represents "surface energy." Denote by γ_i ($i = 1, 2, \dots, n$) component segments of Γ_i which correspond to e_i . In this study we are concerned with the following problem:

Problem P. Find an admissible Γ_G for G that minimizes

$$(1) \quad E[\Gamma_G] = \sum_{i=1}^n \sigma_i |\gamma_i|,$$

where $|\gamma_i|$ denote the lengths of γ_i .

This problem arises in grain boundary motions of annealing pure metal. Critical points of $E[\Gamma_G]$ represent stationary states of a curvature-driven motion, which models the grain boundary motions. A curvature-driven motion with a triple junction has been introduced by Mullins [6]. Later, the motion was derived formally by Bronsard and Reitich [1] as the singular limit of a vector-valued Allen-Cahn equation. Bronsard and Reitich [1] also showed short-time existence of the motion. Let $\Gamma_i(t)$ ($i = 1, 2, 3$) represent curves at time $t > 0$ contained in a two-dimensional bounded region Ω with smooth boundary $\partial\Omega$. Suppose $\Gamma_i(t)$ ($i = 1, 2, 3$) meet at one point $m(t)$. The evolving interface that we consider is subject to the following laws:

(M1) The normal velocity of the interface is given by its curvature.

(M2) At the triple junction $m(t)$, the contact angle θ_k between $\Gamma_i(t)$ and $\Gamma_j(t)$ is given by Young's law, where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$. That is, for positive constants $\sigma_1, \sigma_2, \sigma_3$,

$$\frac{\sin \theta_1}{\sigma_1} = \frac{\sin \theta_2}{\sigma_2} = \frac{\sin \theta_3}{\sigma_3},$$

where $0 < \theta_k < \pi$ and $\theta_1 + \theta_2 + \theta_3 = 2\pi$.

(M3) At the other end of each curve, $\Gamma_i(t)$ touches $\partial\Omega$ at the right angle.

The interfaces have Energy $E(t)$, which decreases as time goes:

$$E(t) = \sigma_1 |\Gamma_1(t)| + \sigma_2 |\Gamma_2(t)| + \sigma_3 |\Gamma_3(t)|,$$

where $|\Gamma_i(t)|$ ($i = 1, 2, 3$) mean the lengths of curves $\Gamma_i(t)$. Stationary interfaces of the motion can be viewed as critical points of the energy. In this connection Sternberg and Ziemer [7] have proved the existence of local minimizers of the energy in clover-like regions. Here we remark that stationary interfaces consist of straight line segments.

On the other hand, Ikota and Yanagida [4] have studied stabilities of stationary interfaces of the motion (M1)–(M3) by linearizing corresponding equations around the stationary interfaces. They linearized the equations formally and analyzed the resulting elliptic operator rigorously. Later they have extended their results to stationary interfaces of binary-tree type with more than one triple junctions[5]. The results are stated as follows.

Theorem 1.1. Let $\Gamma = \{\gamma_i\}$ be a stationary interface that is homeomorphic to a binary tree. Denote by L_i the length of γ_i . Define a characteristic index D by

$$D = \sum_{\gamma_i \in \Gamma} \sigma_i L_i \times \prod_{\gamma_i \in B} h_i + \sum_{\gamma_i \in B} \left\{ \sigma_i \prod_{\gamma_j \in B \setminus \{\gamma_i\}} h_j \right\},$$

where h_i denotes the curvature of $\partial\Omega$ at the point of contact with $\gamma_i \in B$. (Note that h_i is taken to be nonpositive if Ω is convex.)

(i) The unstable dimension N_U is given by

$$N_U = \begin{cases} m - 1 & \text{for } (-1)^m D < 0, \\ m & \text{for } (-1)^m D > 0, \end{cases}$$

where $m = \#\{h_i < 0\}$.

(ii) The stationary interface is degenerate (i.e., there exists a zero eigenvalue) if and only if $D = 0$.

We remark that the index D is independent of the topology of Γ .

Although Ikota and Yanagida have established a stability criterion assuming the existence of stationary interfaces, it has not been known whether given regions have stationary interfaces in general. The existence problem can be regarded as a variation of the Fermat-Steiner problem[2].

In [4] and [5], stabilities of stationary states have been studied on the assumption of the existence of stationary states.

In the present study we show that stationary states do exist for convex Ω . Our problem can be regarded as a variant of the Fermat-Steiner problem, though the treatments are quite different.

The Fermat-Steiner problem is described as follows: for a given triangle $\triangle ABC$, find a point P that minimizes the sum of lengths

$$|PA| + |PB| + |PC|.$$

This problem was proposed by Fermat to Torricelli. Afterwards Steiner considered the same problem and gave a systematic solution. In [2] Gueron and Tessler solved the weighted Fermat-Steiner problem. They also gave an interesting historical survey of the problem.

Now we are in a position to state our result.

Theorem 1.2. *Suppose Ω is convex. Let n be a positive integer and G a binary tree with n triple junctions. Then there exists at least one critical interface of E which is admissible for G .*

2 Outline of Proof

Before proceeding with Problem P, we consider the two phase separation problem with no triple junctions as an illustration. Let Ω be a convex domain in \mathbb{R}^2 . Suppose two points P_1 and P_2 are on the boundary $\partial\Omega$. We seek a critical interface of $E(P_1P_2) = |P_1P_2|$, the length of a line segment P_1P_2 .

A simple calculation shows that P_1P_2 is critical if and only if P_1P_2 intersects with $\partial\Omega$ at the right angle. Thus all we have to do is to find P_1P_2 such that P_1P_2 are orthogonal to $\partial\Omega$ at both P_1 and P_2 .

We parameterize $\partial\Omega$ by an arc length parameter s : $s \mapsto P(s) = (x(s), y(s)) \in \partial\Omega$. By $\tau(s)$ we denote the tangential vector to $\partial\Omega$ at $P(s)$, that is $\tau(s) = (\partial/\partial s)(x(s), y(s))$. For any point $P_1 = P(s_1) \in \partial\Omega$, we can choose $s = s_2$ so that $\tau(s_2)$ is parallel to $\tau(s_1)$.

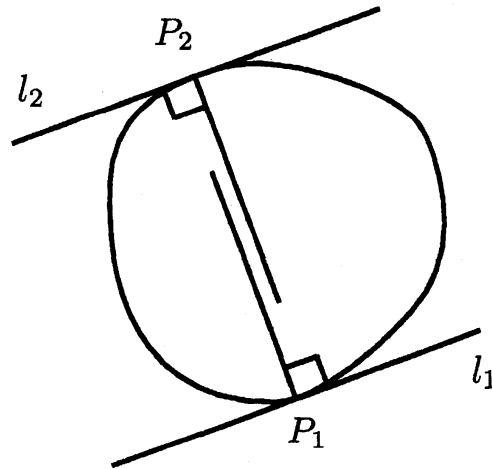


Figure 2: Lines l_1 and l_2 are rotated along $\partial\Omega$.

Then we move s_1 and observe variations of the distance $d(l_1, l_2)$, where l_i are tangential lines to $\partial\Omega$ at $P(s_i)$ ($i = 1, 2$). We can easily see that the distance $d(l_1, l_2)$ is critical if and only if P_1P_2 intersects with $\partial\Omega$ orthogonally. Since $d(l_1, l_2)$ has a maximum (and a minimum), the energy $E(P_1P_2)$ has a critical interface.

Now we turn our attention to Problem P. We consider the case where G has a single triple junction; for more triple junctions we can show the existence of critical interfaces

by induction. We can easily verify that $\Gamma_G = (\gamma_1, \gamma_2, \gamma_3)$ is critical if and only if the following two conditions are satisfied:

1. $\angle(\gamma_i, \gamma_j) = \theta_k \quad (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$.
2. $\gamma_i \perp \partial\Omega \quad (i = 1, 2, 3)$.

Let $\nu(s_i)$ be the unit normal to $\partial\Omega$ at $P(s_i)$ pointing inside Ω . Likewise in the analysis of the two phase problem, we can choose s_2 and s_3 for s_1 such that

$$\angle(\nu(s_i), \nu(s_j)) = \theta_k, \quad (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2).$$

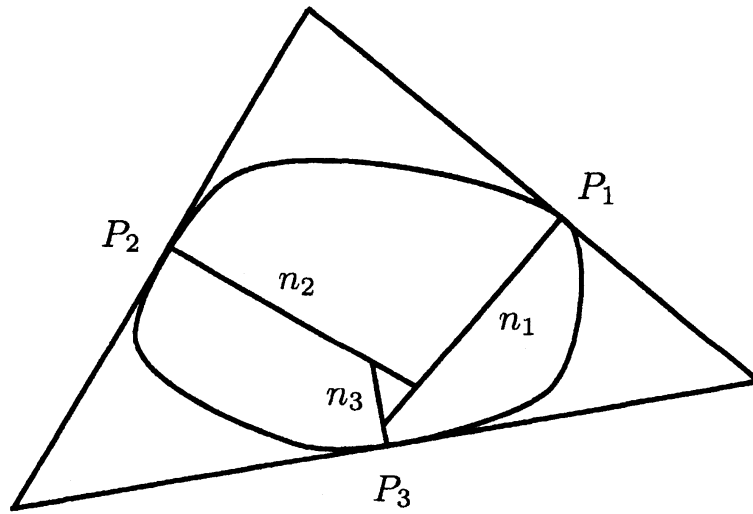


Figure 3: Lines n_1, n_2, n_3 are rotated.

Let \mathcal{T} be the triangle composed of l_1, l_2, l_3 , where l_i are again the tangential lines at $P_i = P(s_i)$, and $T(s)$ the area of \mathcal{T} . Denote by n_i the normal line to $\partial\Omega$ at P_i . Then we can prove that n_1, n_2, n_3 meet at one point if and only if $\partial T / \partial s = 0$. This indicates that the n_i ($i = 1, 2, 3$) make a critical Γ_G .

3 Concluding Remarks

If Ω is not convex, the approach we took in the previous section does not work in general. We illustrate it in the two phase problem.

Let a, b be positive constants. We introduce two graphs in \mathbb{R}^2 :

$$y = g_1(x) = (x - a)^3,$$

$$y = g_2(x) = x^3 + b.$$

Suppose $\partial\Omega$ is represented by $g_1(x)$ and $g_2(x)$ locally. We parameterize the two parts as $(\xi, (\xi - a)^3)$ and $(-\xi, -\xi^3 + b)$ respectively. Here ξ runs over some interval $(-\delta, \delta)$. Then the distance between l_1 and l_2 are given by

$$d(l_1, l_2) = (4\xi^3 + 3a\xi^2 + b)/\sqrt{9\xi^4 + 1}.$$

Straightforward calculation shows that

$$\left. \frac{\partial}{\partial \xi} d(l_1, l_2) \right|_{\xi=0} = 0.$$

However the two normal lines at $\xi = 0$ do not coincide.

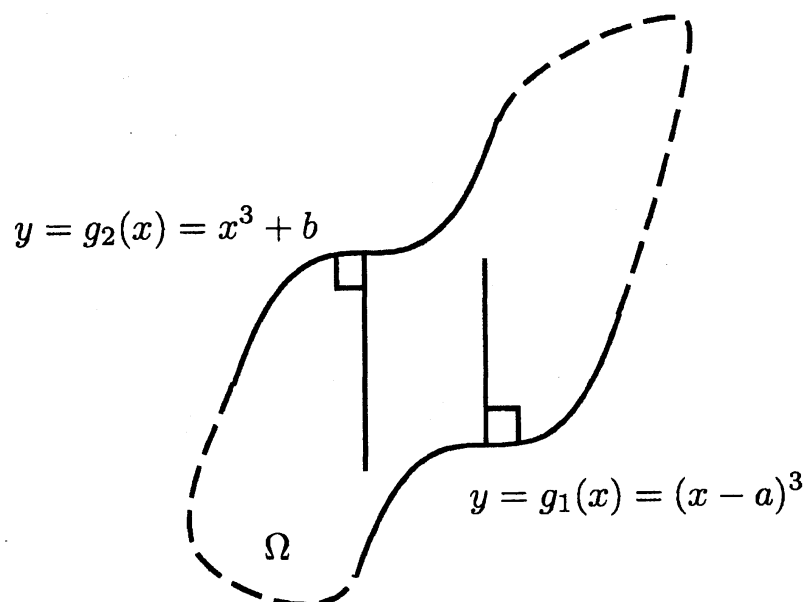


Figure 4: The critical lines of the distance $d(l_1, l_2)$ do not coincide.

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